Common errors in the use of the Stefan-Boltzmann equation

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ABSTRACT

This paper identifies technical errors in the use of the Stefan-Boltzmann equation in current climate research. Analysis was carried out for several statements (conclusions) that were derived from the Stefan-Boltzmann equation to demonstrate how these technical errors can affect our understanding and interpretation of the earth climate system.
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1. INTRODUCTION

Climate scientists frequently make technical errors in their use of the Stefan-Boltzmann equation.

The Stefan-Boltzmann equation is simple: a black-body object with surface temperature, *T*, emits energy per unit time and unit surface area, *J*, the energy flux density:

$$J = \sigma T^4 \qquad (1)$$

where σ is the Stefan-Boltzmann constant equal to 5.67 x 10^{-8} (W/m^2K^4).

When the Stefan-Boltzmann law is applied to the Earth-Atmosphere system, climate scientists often make one or more of these technical errors:

i) a coefficient ε in the range 0 to 1, called emissivity should multiply the right hand side, but not when applied to objects that are not black bodies;
ii) a failure to specify correctly the “surface” and “surface temperature” of the Earth-Atmosphere system;
iii) a failure to specify whether or not a layer of air is a single object or a cluster of objects.

These errors can be easily demonstrated by examining several statements (methodologies) most popular in current climate research:

1) the 33°C greenhouse warming effect for the Earth;
2) the 390 W/m^2 surface radiation in the Earth Energy Budget;
3) the 1˚C CO_2 non-feedback climate sensitivity; and
4) the formula for emission by a layer of air.

2. THEORETICAL ANALYSIS

2.1 The 33°C greenhouse warming effect for the Earth

It is repeatedly stated that the average temperature of the Earth would be 33°C lower than today if there were no greenhouse gas warming effect [1-5]. 33°C = 15°C – (-18°C). The -18°C is obtained from equation of radiative equilibrium between the incoming flux from the Sun and the outgoing flux from the Earth:

$$\pi r^2 (1-\alpha) S_0 = 4\pi r^2 \varepsilon \sigma T^4 \qquad (2)$$

where r is the radius of the Earth, α is the albedo of the Earth, and S_0 (=1368 W/m^2) is the solar constant representing the incoming solar radiation energy flux density. The symbol ε is emissivity of the earth surface.

In much current climate research, ε is either missing or assumed to be 1. Inserting the values of $\alpha = 0.3$ and $\varepsilon = 1$ into Eq. (2), and solving for T:

$$T = \sqrt[4]{\frac{(1-\alpha)S_0}{4\varepsilon\sigma}} = \sqrt[4]{\frac{(1-0.3)\times 1368}{4\times 5.67\times 10^{-8}}} = 254.9\,(\mathrm{K}) \cong 255\,(\mathrm{K}) \cong -18°\mathrm{C} \qquad (3)$$

By adopting $\varepsilon = 1$, however, we are assuming that the earth's surface is a black-body surface, which is never true. If ε is not 1, but 0.9, 0.8, 0.7 or 0.6, T would be -11.4°C, -3.6°C, 5.5°C or 16.5°C respectively. The finding of -18°C is simply a result of a technical error. In fact, the emissivity of the earth surface can be determined $\varepsilon \approx 0.7$ from satellite outgoing radiation spectra.

The Earth's mean near-surface air temperature, as measured by global weather stations, is around 15°C (≅ 288K). N_2 and O_2, which are literally transparent bodies, constitute 99% of the air. This 15°C near surface air temperature is simply a different physical quantity that can not be used to subtract -18°C. White and transparent bodies emit nothing at any temperature.

This error originates from a misunderstanding of the word "surface" that is a symbolised conception of the Stefan-Boltzmann law. If there is no atmosphere, the surface means the land and water ground surface of the Earth, and T represents the mean temperature of the ground surface. If there is atmosphere that is all of nitrogen and oxygen, the surface is still the ground surface, and T still the mean temperature of the ground surface, regardless what the air temperature may be. This is because nitrogen and oxygen are non-radiative (literally $\varepsilon = 0$ for transparent and white bodies). 0 multiplying anything leads to 0.

When we identify the whole Earth-Atmosphere system as an object, its surface and surface temperature are no longer straightforward, but have different values for different radiation wavelengths. Over the absorption bands of water vapour and carbon dioxide (e.g. the absorption band 15 μm for CO_2), the surface is a layer of air starting from the top of atmosphere (TOA) with thickness equal to absorption depth, while the "surface temperature" is the mean temperature of CO_2 molecules within the air layer (≅ -50°C). Similarly one can discover the surface and surface temperature for any other absorbing bands of radiative gases. For the rest of bands, the surface and surface temperature are the ground surface and its mean temperature (≅ 12°C) [e.g. Figure 3 in ref. 6, 7]. What T stands for in Eq. (2) is the mean value of the "surface temperatures" for each wavelength averaged in terms of radiation over all the wavelengths.

2.2 The 390 W/m^2 surface radiation in the Earth Energy Budget

Figure 1 is a diagram shown in the IPCC fourth report (AR4) as an estimate of the Earth's annual and global mean energy balance [8-13].

We examine the surface radiation 390 W/m^2 leaving the earth ground surface, which is considered to correspond to a blackbody emission, p, at 15°C as per the Stefan-Boltzmann equation (1):

$$p = \sigma T^4 = 5.67 \times 10^{-8} \times (273.15 + 15)^4 = 390.89 \cong 390 \text{ (W/m}^2\text{)} \qquad (4)$$

Firstly, the earth ground is never a black-body. Emissivity for the earth ground surface, ε_g, is omitted in Eq. (4).

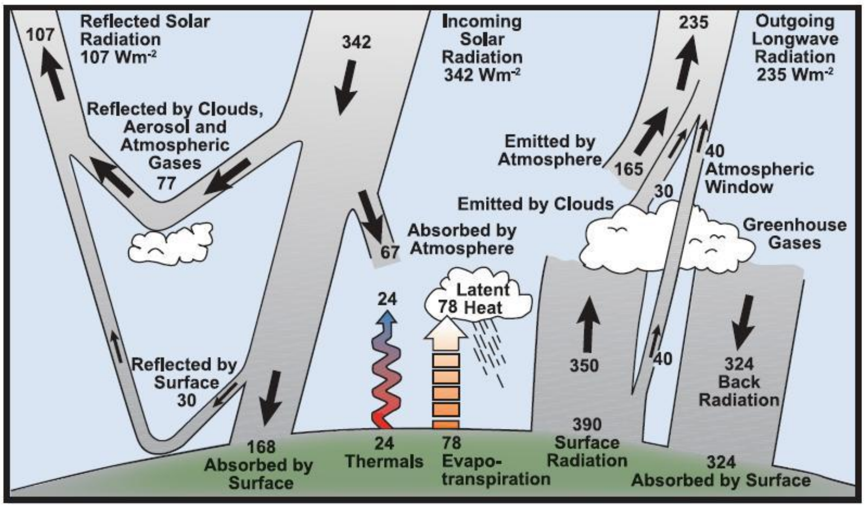

Figure 1. Earth energy budget diagram of IPCC report AR4 2007

Secondly, the Earth's mean near-surface air temperature 15°C has been used. N_2 and O_2 emit literally nothing at whatever temperatures. The T in Eq. (4) must be the temperature of the earth ground surface, which is 285.04 K (11.89°C) [6, 7], as determined from outgoing spectroscopy measurements and simulations. The ground surface radiation then reads:

$$p = \varepsilon_g\, \sigma T^4 = \varepsilon_g \times 5.67 \times 10^{-8} \times 285.04^4 = \varepsilon_g \times 374.29 \text{ (W/m}^2\text{)} \qquad (5)$$

The emissivity of the earth ground surface, ε_g, is unlikely close to 1.0. Black body is an abstraction of a physical concept, hardly any substance is a black body on the Earth.

One can easily understand why the ground surface of the Earth would not completely absorb the 324 W/m^2 back radiation because it is never a black body surface. As these two figures are wrong, many other figures shown on the earth emission tree are called into question.

2.3 The 1˚C CO_2 non-feedback climate sensitivity

It is well accepted in current climate research that a doubling of CO_2 by itself contributes about 1°C to greenhouse warming, known as CO_2 non-feedback climate sensitivity, or CO_2 direct climate sensitivity [14, 15]. The debate is about feedback; a positive feedback will lead to higher, a negative feedback to lower, overall climate sensitivity.

Let us examine how this statement has been derived. The energy emitted by the Earth-Atmosphere system per unit time and unit surface area (radiative flux, aka forcing), *F*, is written:

$$F = \sigma T^4 \tag{6}$$

The derivative of *F* with respect to *T* reads:

$$\frac{dF}{dT} = 4\sigma T^3 \tag{7}$$

Therefore,

$$\Delta T \approx \frac{1}{4\sigma T^3} \ \Delta F \tag{8}$$

Eq. (8) has been interpreted to indicate how much warming ΔT occurs for any forcing increment. If CO_2 doubles, ΔF has been determined as 3.7 W/m^2 by spectroscopic study. Inserting ΔF =3.7 W/m^2, T = 255 K into Eq. (8) leads to:

$$\Delta T \approx \frac{1}{4\sigma T^3} \ \Delta F = \frac{1}{4 \times 5.67 \times 10^{-8} \times 255^3} \times 3.7 \ = 0.984°\text{C} \approx 1°\text{C}$$

Derivation including the emissivity reads,

$$. \ \Delta T \approx \frac{1}{4\varepsilon\sigma T^3} \ \Delta F \tag{9}$$

Taking advantage of the relationship between T and S_0 in Eq. (2), one obtains:

$$\Delta T \approx \frac{1}{4\varepsilon\sigma T^3} \ \Delta F = \frac{T}{(1-\alpha)S_0} \Delta F = \frac{T}{(1-0.3)\times 1368} \times 3.7 = 0.003864 \times T \tag{10}$$

Inserting T = 255 K into Eq. (10) leads to the same answer of 0.985°C. Note that ΔT depends on the emissivity ε via T, even though ε is not explicit in Eq. (10). If ε is 1.0, 0.9, 0.8, 0.7 or 0.6, T would be -18°C, -11.4°C, -3.6°C, 5.5°C or 16.5°C respectively (as above), and, by Eq. (10), ΔT would be 0.98°C, 1.01°C, 1.04°C, 1.07°C or 1.11°C respectively.

The error resulting from omission of emissivity thus cannot be more than 10%; it is more a methodological issue in this case. The problem is, however, that the temperature T, is a physical quantity different from the Earth's mean near-surface air temperature, $T_{air}(h)$, which is largely the temperature of N_2 and O_2 that are literally transparent bodies emitting nothing at whatever temperatures. The symbol h denotes altitude, almost 0 for near surface. To estimate CO_2 direct climate sensitivity, one must seek the relationship between $\Delta T_{air}(h) \sim \Delta F$, not $\Delta T \sim \Delta F$. There are heat transfer mechanisms other than radiation linking this T and $T_{air}(h)$.

All the parameters must be consistent with the object defined either explicitly or implicitly. The CO_2 radiative forcing ΔF =3.7 W/m^2 is actually the forcing of absorption by CO_2 molecules in the atmosphere. The outgoing forcing that leaves the Earth-Atmosphere object remains unchanged while doubling CO_2, as long as the solar constant and albedo are unchanged.

2.4 The formula for emission by a layer of air

The atmosphere is from time to time represented by a layer (or layers) of air for climate modelling [8]. Consider a given layer of air with temperature, T_a, and surface area, S, as shown in Figure 2. In current climate research the Stefan-Boltzmann Equation is straightforwardly applied to obtain σT_a^4 for emitting energy flux density of the air layer. It is treated just like a sheet of solid (or condensed matter) object.

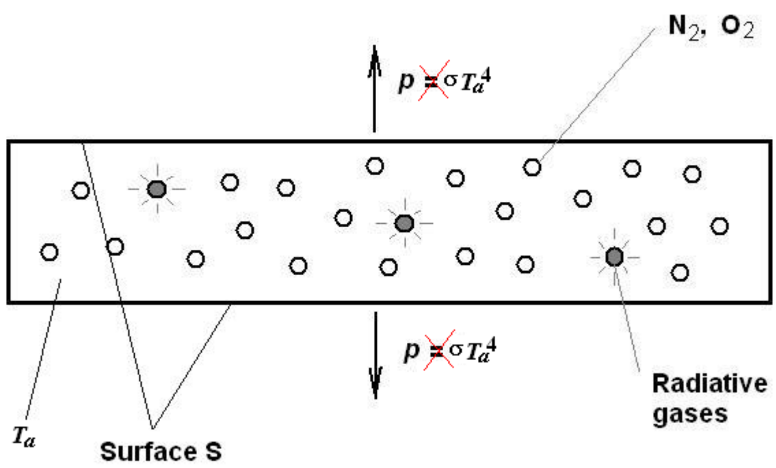

Figure 2 A layer of air can not be treated as a layer of solid object to calculate the emitting power.

As discussed above, N_2 and O_2 do not emit at any temperature. Only the radiative gases in the air layer emit. One will easily realise that i) only the temperature of radiative species is relevant instead of the average temperature of the layer – different gases may have different temperatures due to different radiation properties; ii) no gas is a black body, even the radiative gases.

There is a fundamental methodological error here. Because the emitting species are so sparse in air, a given layer of air can not be identified as a single object applicable to the Stefan-Boltzmann law (strictly speaking, Planck's law). The correct methodology is to identify each single radiative molecule as an object that emits according to its temperature and radiative bands, forming a cluster of objects within the layer of air. How much the layer of air emits must be determined by summing up all the radiation energy density emitted by each individual molecular object upon the surface S. The principle of formulation is shown in a simple example as follows.

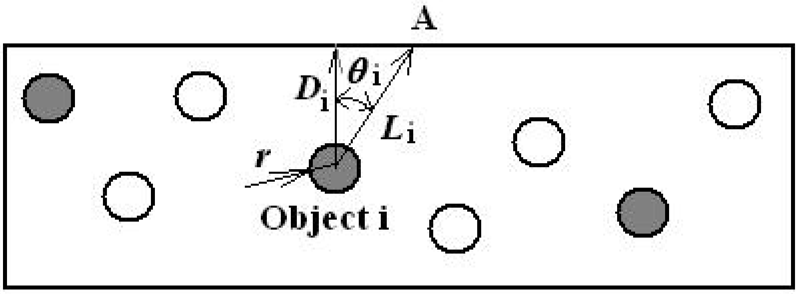

Figure 3 Geometrical parameters defined for derivation of Eq. (11).

Assume a layer of air containing n tiny spherical grey body objects with uniform radius, r, emissivity, ε, and uniform temperature T, the distance from each object to a given point on the surface of the air layer L_i, where i is from 1 to n, as shown in Figure 3. On the surface of each individual object, the emission flux density (energy per unit area and unit time) must follow the Stefan-Boltzmann equation, i.e. $\varepsilon\,\sigma T^4$. The flux density decays with distance L_i to $(r/L_i)^2\,\varepsilon\,\sigma T^4$. Therefore, at a given point on the surface of the air layer the overall flux density, p, reads:

$$p=\sum_{i=1}^{n}\left(\left(\frac{r^2}{L_i^2}\cos\theta_i\right)\varepsilon\sigma T^4\right)=\left(r^2\sum_{i=1}^{n}\frac{D_i}{L_i^3}\right)\varepsilon\,\sigma T^4=\zeta\,\varepsilon\sigma T^4 \qquad (11)$$

where,

$$\zeta=r^2\sum_{i=1}^{n}\frac{D_i}{L_i^3} \qquad (12)$$

is another coefficient ($0 < \zeta <= 1$), which is missing in the current climate research. Note $\zeta = 0$ for $n = 0$, and ζ approaches 1 as n is sufficiently large enough in a given volume. This coefficient applies as well for the Planck distribution function.

Eq. (11) indicates the flux density is very much depends on the number of objects within the air layer. If the radiative objects are not dense enough within the air layer, the term of summation will be a very small fraction. A strict mathematical expression can be formulised along this line but omitted in this article.

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3. CONCLUSION

There is no surprise that scientists can make errors, but it is perhaps a surprise that the technical errors have been shared by so many scientists across a discipline to such an unprecedented extent. Hopefully this article, by illuminating these errors, will help to advance climate science.

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